

A GLOBAL BRIANÇON-SKODA-HUNEKE-SZNAJDMAN THEOREM

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ABSTRACT. We prove a global effective membership result for polynomials on a non-reduced algebraic subvariety of \mathbb{C}^N . It can be seen as a global version of a recent local result of Sznajdman, generalizing the Briançon-Skoda-Huneke theorem for the local ring of holomorphic functions at a point on a reduced analytic space.

1. INTRODUCTION

Let x be a point on a smooth analytic variety X of pure dimension n and let \mathcal{O}_x be the local ring of holomorphic functions. The classical Briançon-Skoda theorem, [11], states that if $(a) = (a_1, \dots, a_m)$ is any ideal in \mathcal{O}_x and ϕ is in \mathcal{O}_x , then $\phi \in (a)^r$ if

$$(1.1) \quad |\phi| \leq C|a|^{\nu+r-1}$$

holds with $\nu = \min(m, n)$. The proof given in [11] is purely analytic. However, the condition (1.1) is equivalent to saying that ϕ belongs to the integral closure $\overline{(a)^{\nu+r-1}}$, and thus the theorem admits a purely algebraic formulation. Therefore it was somewhat astonishing that it took several years before algebraic proofs were found, [22, 23]. Later on, Huneke, [19], proved a far-reaching algebraic generalization which contains the following statement for non-smooth X .

Let $x \in X$ be a point on a reduced analytic variety of pure dimension. There is a number ν such that if $(a) = (a_1, \dots, a_m)$ is any ideal in \mathcal{O}_x and ϕ is in \mathcal{O}_x , then (1.1) implies that $\phi \in (a)^r$.

An important point is that ν is uniform with respect to both (a) and r . The smallest possible such ν is called the Briançon-Skoda number, and it depends on the complexity of the singularities of X at x . An analytic proof of this statement appeared in [3]. A nice variant for a non-reduced X of pure dimension is formulated and proved in [28].

Let x be a point on a non-reduced analytic space X of pure dimension n , and let X_{red} be the underlying reduced space, cf., Section 2.1 below. There is a natural surjective mapping $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X_{red},x}$. Let $i: X \rightarrow \Omega \subset \mathbb{C}^N$ be a local embedding, and let $\mathcal{J}_{X,x}$ be the associated local ideal in $\mathcal{O}_{\Omega,x}$, so that $\mathcal{O}_x = \mathcal{O}_{X,x} = \mathcal{O}_{\Omega,x}/\mathcal{J}_{X,x}$. A holomorphic differential operator L in Ω is Noetherian at x if $L\phi$ vanishes on $X_{red,x}$ (or equivalently, $L\phi \in \sqrt{\mathcal{J}_{X,x}} = \mathcal{J}_{X_{red},x}$) for all $\phi \in \mathcal{J}_{X,x}$. Such an L defines an intrinsic mapping

$$L: \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X_{red},x}, \quad \phi \mapsto L\phi.$$

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Theorem 1.1 (Sznajdman, [28]). *Given $x \in X$, there is a finite set L_α of Noetherian operators at x and a number ν such that for each ideal $(a) = (a_1, \dots, a_m) \subset \mathcal{O}_{X,x}$ and $\phi \in \mathcal{O}_{X,x}$,*

$$(1.2) \quad |L_\alpha \phi| \leq C|a|^{\nu+r} \text{ on } X_{red,x}$$

for all α , implies that $\phi \in (a)^r$.

Here $|a|$ means $|a_1| + \dots + |a_m|$ (where $|a_j|$ is the modulus of the image of a_j in $\mathcal{O}_{X,x}$), which up to constants is independent of the choice of generators of the ideal (a) . The condition (1.2) means that $L_\alpha \phi$ is in the integral closure of the image in $\mathcal{O}_{X_{red,x}}$ of $(a)^{\nu+r}$.

Applying to $(a) = (0)$ we find that $L_\alpha \phi = 0$ on $X_{red,x}$ for all α implies that $\phi = 0$ in $\mathcal{O}_{X,x}$.

We now turn our attention to global variants. Let V be a purely n -dimensional algebraic subvariety of \mathbb{C}^N and let $J_V \subset \mathbb{C}[x_1, \dots, x_N]$ be the associated ideal. Assume that F_j are polynomials in \mathbb{C}^N of degree $\leq d$. If the polynomial Φ belongs to the restriction of the ideal (F_1, \dots, F_m) to V , i.e., there are polynomials Q_j such that

$$(1.3) \quad \Phi = \sum_{j=1}^m F_j Q_j + J_V,$$

then it is natural to ask for a representation (1.3) with some control of the degree of Q_j . It is well-known that if $V = \mathbb{C}^N$, then in general $\max_j \deg F_j Q_j$ must be doubly exponential in d , i.e., like 2^{2^d} . However, in the Nullstellensatz, i.e., $\Phi = 1$, then (roughly speaking) d^n is enough, this is due to Kollár, [21], and Jelonek, [20]. In [18] Hickel proved a global effective version of the Briançon-Skoda theorem for polynomial ideals in \mathbb{C}^n , basically saying that if $|\Phi|/|F|^{\min(m,n)}$ is locally bounded, then there is a representation (1.3) in \mathbb{C}^n with $\deg F_j Q_j \leq \deg \Phi + Cd^n$. For the precise statement, see [18] or [7]. In [7, Theorem A] a generalization to polynomials on reduced algebraic subvarieties of \mathbb{C}^N appeared. Our objective in this paper is to find a generalization to a not necessarily reduced algebraic subvariety V of \mathbb{C}^N of pure dimension n .

Let X be the closure (see Section 2.2) of V in \mathbb{P}^N and let X_{red} be the underlying reduced variety. Given polynomials F_1, \dots, F_m , let f_j denote the corresponding d -homogenizations, considered as sections of the line bundle $\mathcal{O}(d)|_{X_{red}}$, and let \mathcal{J}_f be the coherent analytic sheaf on X_{red} generated by f_j . Furthermore, let c_∞ be the maximal codimension of the so-called *distinguished varieties* of the sheaf \mathcal{J}_f , in the sense of Fulton-MacPherson, that are contained in

$$X_{red,\infty} := X_{red} \setminus V_{red},$$

see Section 5. It is well-known that the codimension of a distinguished variety cannot exceed the number m , see, e.g., [13, Proposition 2.6], and thus

$$c_\infty \leq \min(m, n).$$

We let Z_f denote the zero variety of \mathcal{J}_f in X_{red} .

Let $\text{reg } X$ denote the so-called (*Castelnuovo-Mumford*) *regularity* of $X \subset \mathbb{P}^N$, see Section 2.2 below. We can now formulate the main result of this paper.

Theorem 1.2 (Main Theorem). *Assume that V is an algebraic subvariety of \mathbb{C}^N of pure dimension n and let X be its closure in \mathbb{P}^N . There is a finite set of holomorphic differential operators L_α on \mathbb{C}^N with polynomial coefficients and a number ν so that the following holds:*

(i) For each point $x \in V$ the germs of L_α are Noetherian operators at x such that the conclusion in Theorem 1.1 holds.

(ii) If F_1, \dots, F_m are polynomials of degree $\leq d$, Φ is a polynomial, and

$$(1.4) \quad |L_\alpha \Phi|/|F|^\nu \text{ is locally bounded on } V_{red}$$

for each α , then there are polynomials Q_1, \dots, Q_m such that (1.3) holds and

$$(1.5) \quad \deg(F_j Q_j) \leq \max(\deg \Phi + \nu d^{c_\infty} \deg X_{red}, (d-1) \min(m, n+1) + \operatorname{reg} X).$$

If there are no distinguished varieties of \mathcal{J}_f contained in $X_{red, \infty}$, then d^{c_∞} shall be interpreted as 0.

In case V is reduced we can choose L_α as just the identity; then (ii) is precisely (part (i) of) Theorem A in [7]. If $V = \mathbb{C}^n$ we get back Hickel's theorem, [18], mentioned above.

Example 1.3. If we apply Theorem 1.2 to Nullstellensatz data, i.e., F_j with no common zeros on V and $\Phi = 1$, then the hypothesis (1.4) is fulfilled, and we thus get Q_j such that $F_1 Q_1 + \dots + F_m Q_m - 1$ belongs to J_V and

$$\deg(F_j Q_j) \leq \max(\nu d^{c_\infty} \deg X_{red}, (d-1) \min(m, n+1) + \operatorname{reg} X).$$

See [7, Section 1] for a discussion of this estimate in the reduced case. \square

Example 1.4. If f_j have no common zeros on X and Φ is any polynomial, then there is a solution to (1.3) such that

$$\deg F_j Q_j \leq \max(\deg \Phi, (d-1)(n+1) + \operatorname{reg} X).$$

If $X = \mathbb{P}^n$, then $\operatorname{reg} X = 1$ and so we get back the classical Macaulay theorem. \square

Remark 1.5. It follows that L_α is a set of Noetherian operators such that a polynomial $\Phi \in \mathbb{C}[x_1, \dots, x_N]$ is in $J_V \subset \mathbb{C}[x_1, \dots, x_N]$ if and only if $L_\alpha \Phi = 0$ on V_{red} for each α . The existence of such a set is well-known, and a key point in the celebrated Ehrenpreis-Palamodov fundamental theorem, [16] and [25]; see also, e.g., [9] and [24]. \square

Remark 1.6. It turns out, see Theorem 4.1 below, that the Noetherian operators L_α in Theorem 1.2 have the following additional property: For each α there is a finite set of holomorphic differential operators $M_{\alpha, \gamma}$ such that

$$(1.6) \quad L_\alpha(\Phi \Psi) = \sum_{\gamma} L_\gamma \Phi M_{\alpha, \gamma} \Psi$$

for any holomorphic functions Φ and Ψ . This formula shows that set of functions that satisfy (1.2) at a point x is indeed an ideal. \square

By homogenization, this kind of effective results can be reformulated as geometric statements: Let $z = (z_0, \dots, z_N)$, $z' = (z_1, \dots, z_N)$, let $f_i(z) := z_0^d F_i(z'/z_0)$ be the d -homogenizations of F_i , considered as sections of $\mathcal{O}(d) \rightarrow \mathbb{P}^N$, and let $\varphi(z) := z_0^{\deg \Phi} \Phi(z'/z_0)$. Then there is a representation (1.3) on V with $\deg(F_j Q_j) \leq \rho$ if and only if there are sections q_i of $\mathcal{O}(\rho - d)$ on \mathbb{P}^N such that

$$(1.7) \quad f_1 q_1 + \dots + f_m q_m = z_0^{\rho - \deg \Phi} \varphi$$

on X in \mathbb{P}^N ; that is, the difference of the right and the left hand sides belongs to the sheaf \mathcal{J}_X .

To prove Theorem 1.2 we first have to define a suitable set of global Noetherian operators on \mathbb{P}^N . This is done in Section 4 following the ideas of Björk, [10], in

the local case, starting from a representation of \mathcal{J}_X as the annihilator of a tuple of so-called Coleff-Herrera currents on \mathbb{P}^N . The rest of the proof of Theorem 1.2, given in Section 5, follows to a large extent the proof of Theorem A in [7]. By the construction in [5] we have a residue current R^X associated with \mathcal{J}_X such that the annihilator ideal of R^X is precisely \mathcal{J}_X . Following the ideas in [7] we then form the “product” $R^f \wedge R^X$, where R^f is the current of Bochner-Martinelli type introduced in [1], inspired by [26]. By computations as in [28], the condition (1.4) ensures that ϕ annihilates this current at each point $x \in V_{red}$. If ρ is large enough, this is reflected by the first entry of the right hand side of (1.5), then a geometric estimate from [13] ensures that the ρ -homogenization ϕ of Φ indeed satisfies a condition like (1.4) even at infinity. Therefore ϕ annihilates the current $R^f \wedge R^X$ everywhere on \mathbb{P}^N . For this argument it is important that the Noetherian operators extend to \mathbb{P}^N . The proof of Theorem 1.2 is then concluded along the same lines as in [7] by solving a sequence of $\bar{\partial}$ -equations. If ρ is large enough, this is reflected by the second entry in the right hand side of (1.5), there are no cohomological obstructions. We then get a global representation of ϕ as a member of $\mathcal{O}(\rho) \otimes (\mathcal{J}_f + \mathcal{J}_X)$. After dehomogenization we get the desired representation (1.3).

In Section 2 we collect some necessary background material. In Section 3 we discuss global Coleff-Herrera currents on projective space. As mentioned above, the proof of our main theorem is given in the last two sections.

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2. PRELIMINARIES

In this section we collect various definitions and facts that will be used later on.

2.1. Non-reduced analytic space. A reduced analytic space Z is locally described as an analytic subset of some open set $\Omega \subset \mathbb{C}^N$, and the sheaf \mathcal{O}_Z of holomorphic functions on Z , the *structure sheaf*, is then isomorphic to $\mathcal{O}_\Omega / \mathcal{J}_Z$, where \mathcal{J}_Z is the ideal sheaf of functions in Ω that vanish on Z . A non-reduced analytic space X (also referred to as an analytic scheme) with underlying reduced space Z and *structure sheaf* \mathcal{O}_X is locally of the form $\mathcal{O}_X = \mathcal{O}_\Omega / \mathcal{J}$, where $\mathcal{J} \subset \mathcal{J}_Z$ is a coherent ideal sheaf with common zero set Z . Thus $\mathcal{J}_Z = \sqrt{\mathcal{J}}$ and \mathcal{O}_Z is obtained from \mathcal{O}_X by taking the quotient by all nilpotent elements in \mathcal{O}_X . Given the non-reduced space X we denote the underlying reduced space by X_{red} .

The space X has *pure dimension* n if for each $x \in X_{red}$, all the associated prime ideals of the local ring \mathcal{O}_x has dimension n . In particular, then X_{red} has pure dimension n .

2.2. Algebraic and projective spaces. We will only be concerned with analytic spaces that are globally embedded in some \mathbb{C}^N or \mathbb{P}^N . An analytic subspace $V \subset \mathbb{C}^N$ is *algebraic* if the sheaf \mathcal{J}_V is generated by a finite number of polynomials. Let J_V be the corresponding ideal in the polynomial ring $\mathbb{C}[x_1, \dots, x_N]$. Let J_X be the homogeneous ideal in the graded ring $\mathbb{C}[x_0, \dots, x_N]$ generated by homogenizations of the elements in J_V . If J_V has pure dimension n , then J_X has pure dimension $n + 1$. In particular, 0 is not an associated prime ideal. Each homogeneous polynomial corresponds to a global section of the line bundle $\mathcal{O}(\ell) \rightarrow \mathbb{P}^N$ for some ℓ . These sections define a coherent analytic sheaf \mathcal{J}_X over \mathbb{P}^N of pure dimension n . We define the closure X of V as the analytic subspace of \mathbb{P}^N with structure sheaf $\mathcal{O}_X =$

$\mathcal{O}_{\mathbb{P}^N}/\mathcal{J}_X$. It is clear that the sheaf \mathcal{J}_X coincides with the sheaf \mathcal{J}_V defined by the ideal J_V in \mathbb{C}^N .

Let S be the graded ring $\mathbb{C}[x_0, \dots, x_N]$ and let $S(-d)$ be the S -module that is equal to S but with the gradings shifted by d . Let J_X be the homogeneous ideal in S of all forms that belong to \mathcal{J}_X . Since 0 is not an associated prime ideal of J_X , cf., [14, Corollary 20.14], see also [7, Section 2.7], there is a graded free resolution

$$(2.1) \quad 0 \rightarrow \oplus_1^{r_N} S(-d_N^i) \xrightarrow{c_N} \dots \xrightarrow{c_3} \oplus_1^{r_1} S(-d_1^i) \xrightarrow{c_1} S \rightarrow S/J_X \rightarrow 0$$

of the S -module S/J_X , where $c_k = (c_k^{ij})$ are matrices of homogeneous forms in \mathbb{C}^{N+1} with $\deg c_k^{ij} = d_k^j - d_{k-1}^i$. The number

$$(2.2) \quad \text{reg } X := \max_{k,i} (d_k^i - k) + 1$$

is called the Castenouvo-Mumford regularity of X in \mathbb{P}^N , see, e.g., [15]. This number describes the complexity of the embedding of X in \mathbb{P}^N ; thus two isomorphic analytic spaces embedded in different ways may have different regularities.

2.3. Some residue theory. Let Y be a (smooth) complex manifold of dimension N . Given a holomorphic function f on Y , following Herrera and Lieberman, [17], one can define the principal value current $1/f$ as the limit

$$\lim_{\epsilon \rightarrow 0} \chi(|f|^2 v / \epsilon) \frac{1}{f},$$

where $\chi(t)$ is the characteristic function of the interval $[1, \infty)$ or a smooth approximand and v is any smooth strictly positive function. The existence of this limit for a general f relies on Hironaka's theorem that ensures that there is a modification $\pi: \tilde{Y} \rightarrow Y$ such that $\pi^* f$ is locally a monomial. It is readily checked that $f(1/f) = 1$ and $f\bar{\partial}(1/f) = 0$. The current $1/f$ is well-defined even if f is a holomorphic section of a Hermitian line bundle over Y , since $a(1/af) = 1/f$ if a is holomorphic and nonvanishing.

Example 2.1. In one complex variable it is quite elementary to see that the principal value current $1/s^{m+1}$ exists and that

$$\bar{\partial} \frac{1}{s^{m+1}} \wedge ds \cdot \xi = \frac{2\pi i}{m!} \frac{\partial^m}{\partial s^m} \xi(0),$$

for test functions ξ . □

The sheaf $\mathcal{PM} = \mathcal{PM}_Y$ of *pseudomeromorphic currents*, introduced in [6, 4], consists of currents on Y that are finite sums of direct images under (compositions of) modifications, simple projections and open inclusions of currents of the form

$$\frac{\xi}{s_1^{\alpha_1} \dots s_{\ell-1}^{\alpha_{\ell-1}}} \wedge \bar{\partial} \frac{1}{s_{\ell}^{\alpha_{\ell}}} \wedge \dots \wedge \bar{\partial} \frac{1}{s_m^{\alpha_m}}, \quad m \leq n,$$

in some \mathbb{C}_s^m and ξ is a smooth form with compact support.

The sheaf \mathcal{PM} is closed under $\bar{\partial}$ (and ∂) and multiplication by smooth forms. If τ is in \mathcal{PM} and has support on an analytic subset $V \subset Y$ and η is a holomorphic form that vanishes on V , then

$$(2.3) \quad \bar{\eta} \wedge \tau = 0, \quad d\bar{\eta} \wedge \tau = 0.$$

The first equality roughly speaking means that τ does not involve anti-holomorphic derivatives. By a standard argument the second equality in (2.3) implies:

Dimension principle: If τ is a pseudomeromorphic current on Y of bidegree $(*, p)$ that has support on an analytic subset V of codimension $> p$, then $\tau = 0$.

Let $\mathcal{U} \subset Y$ be an open subset. If τ is in $\mathcal{PM}(\mathcal{U})$ and $V \subset \mathcal{U}$ is an analytic subvariety, then the natural restriction of τ to the open set $\mathcal{U} \setminus V$ has a canonical extension as a principal value to a pseudomeromorphic current $\mathbf{1}_{\mathcal{U} \setminus V} \tau$ on \mathcal{U} . If h is a holomorphic tuple in \mathcal{U} with common zero set V , and χ is a smooth approximand χ of the characteristic function of the interval $[1, \infty)$, then

$$(2.4) \quad \mathbf{1}_{\mathcal{U} \setminus V} \tau = \lim_{\epsilon \rightarrow 0} \chi(|h|^2/\epsilon) \tau.$$

It follows that $\mathbf{1}_V \tau := \tau - \mathbf{1}_{\mathcal{U} \setminus V} \tau$ is pseudomeromorphic in \mathcal{U} and has support on V . Notice that if α is a smooth form, then $\mathbf{1}_V \alpha \wedge \tau = \alpha \wedge \mathbf{1}_V \tau$. Moreover, if $\pi: \tilde{\mathcal{U}} \rightarrow \mathcal{U}$ is a modification, $\tilde{\tau}$ is in $\mathcal{PM}(\tilde{\mathcal{U}})$, and $\tau = \pi_* \tilde{\tau}$, then

$$\mathbf{1}_V \tau = \pi_* (\mathbf{1}_{\pi^{-1}V} \tilde{\tau})$$

for any analytic set $V \subset \mathcal{U}$. For any analytic sets $W, W' \subset \mathcal{U}$,

$$\mathbf{1}_W \mathbf{1}_{W'} \tau = \mathbf{1}_{W \cap W'} \tau.$$

Let $Z \subset Y$ be an analytic subset of pure codimension p and let τ be a pseudomeromorphic current of bidegree $(N, *)$ with support on Z . We say that τ has the *standard extension property*, SEP, with respect to Z if $\mathbf{1}_V \tau = 0$ for each subvariety $V \subset Z \cap \mathcal{U}$ of positive codimension, where $\mathcal{U} \subset Y$ is some open subset. The sheaf of such currents is denoted by \mathcal{W}^Z . If $Z = Y$ we write \mathcal{W} rather than \mathcal{W}^Y . The subsheaf of \mathcal{W}^Z of $\bar{\partial}$ -closed currents of bidegree (N, p) is called the sheaf of Coleff-Herrera currents¹, \mathcal{CH}^Z , on Z .

Remark 2.2. The sheaf \mathcal{CH}^Z was introduced by Björk, in a slightly different way. For the equivalence to the definition given here, see [2, Section 5]. \square

Example 2.3. Let $[Z]$ be the Lelong current associated with Z and let β be a smooth form of bidegree $(p, *)$. Then $\mu = \beta \wedge [Z]$ is in \mathcal{W}^Z . If β is holomorphic, then μ is in \mathcal{CH}^Z . See, e.g., [2, Example 4.2]. \square

Proposition 2.4. *If L is a holomorphic differential operator and τ is in \mathcal{W}^Z , then $\xi \mapsto \tau.L\xi$ defines a current in \mathcal{W}^Z .*

Proof. It is a local statement so by induction it is enough to let L be a partial derivative $\partial/\partial\zeta_1$ with respect to some local coordinate system. Let L denote the Lie derivative with respect to this vector field. Since ξ has bidegree $(0, *)$, $(\partial/\partial\zeta_1)\xi = L\xi$. Thus

$$\tau.(\partial/\partial\zeta_1)\xi = \tau.L\xi = \pm L\tau.\xi,$$

and $L\tau$ is in \mathcal{W}^Z according to [8, Theorem 3.7]. \square

2.4. Almost semi-meromorphic currents. We say that a current b on a smooth manifold Y is *almost semi-meromorphic*, $b \in ASM(Y)$, if there is a modification $\pi: Y' \rightarrow Y$, a holomorphic generically non-vanishing section σ of a line bundle $L \rightarrow Y'$ and an L -valued smooth form ω such that

$$(2.5) \quad b = \pi_* \frac{\omega}{\sigma},$$

¹We adopt here the convention from [10]; in, e.g., [28] these currents have bidegree $(0, p)$.

where ω/σ denotes the principal value current. This class of currents was introduced in [4] and studied in more detail in [8]. All results in this subsection can be found in the latter reference.

Let $ZSS(b)$, the Zariski singular support of b , be the smallest analytic set such that b is smooth in the complement.

We will need the following results.

Proposition 2.5 ([8], Theorem 4.26). *If b is almost semi-meromorphic on Y and L is a holomorphic differential operator, then Lb is almost semi-meromorphic as well.*

Clearly, $ZSS(Lb) \subset ZSS(b)$.

Theorem 2.6 ([8], Theorem 4.8). *If $b \in ASM(Y)$ and τ is any pseudomeromorphic current in Y , then there is a unique current T in Y that coincides with $b \wedge \tau$ outside $ZSS(b)$ and such that $\mathbf{1}_{ZSS(b)}T = 0$.*

We will denote the extension T by $b \wedge \tau$ as well. It follows from (2.4) that

$$(2.6) \quad b \wedge \tau = \lim_{\delta} \chi_{\delta} b \wedge \tau$$

if $\chi_{\delta} = \chi(|g|^2/\delta)$ where g is a holomorphic tuple whose zero set is precisely $ZSS(b)$. It is not hard to check, cf., [8, Proposition 4.9], that if V is any analytic set, then

$$(2.7) \quad \mathbf{1}_V(b \wedge \tau) = b \wedge \mathbf{1}_V \tau.$$

It follows from (2.7) that $b \in ASM(Y)$ induces a mapping

$$\mathcal{W}^Z \rightarrow \mathcal{W}^Z, \quad \tau \mapsto b \wedge \tau.$$

Given $a \in ASM(Y)$ and $\tau \in \mathcal{PM}^Y$ we define

$$\bar{\partial}a \wedge \tau := \bar{\partial}(a \wedge \tau) - (-1)^{\deg a} a \wedge \bar{\partial}\tau$$

The definition is made so that the formal Leibniz rule holds.

Remark 2.7. Clearly $\bar{\partial}a = b + r(a)$ where $b = \mathbf{1}_{X \setminus ZSS(a)} \bar{\partial}a$ and $r(a)$, the residue of a , has support on $ZSS(a)$. One can check, cf., [8, Proposition 4.16], that in fact $b \in ASM(X)$. Thus we can define $r(a) \wedge \tau := \bar{\partial}a \wedge \tau - b \wedge a$. If χ_{δ} is as above, then

$$(2.8) \quad r(a) \wedge \tau = \lim_{\delta} \bar{\partial} \chi_{\delta} a \wedge \tau.$$

□

If a is holomorphic outside $ZSS(a)$, then clearly the support of $\bar{\partial}a \wedge \tau$ is contained in $\text{supp } \tau \cap ZSS(a)$. In particular, if $\gamma_1, \dots, \gamma_p$ are holomorphic functions, then by induction we can form the current

$$(2.9) \quad \bar{\partial} \frac{1}{\gamma_p} \wedge \dots \wedge \bar{\partial} \frac{1}{\gamma_1}.$$

Clearly it is $\bar{\partial}$ -closed and has support on $Z_{\gamma} = \{\gamma_1 = \dots = \gamma_p = 0\}$. If in addition Z_{γ} has codimension p , then (2.9) is anti-commuting in its factors, see, e.g., [6, Section 2]. In this case we call it the *Coleff-Herrera product* μ^{γ} formed by the γ_j . It is well-known, and was first proved by Dickenstein-Sessa and Passare, that the annihilator $\text{ann } \mu^{\gamma} = \{\phi \in \mathcal{O}; \phi \mu^{\gamma} = 0\}$ is precisely equal to the ideal (γ) generated by $\gamma_1, \dots, \gamma_p$, see, [2, Eq. (4.3)] for the setting used here. It follows by the dimension principle that μ^{γ} is in $\mathcal{W}^{Z_{\gamma}}$. If ω is a holomorphic $(N, 0)$ -form, therefore $\mu^{\gamma} \wedge \omega$ is in $\mathcal{CH}^{Z_{\gamma}}$.

Any Coleff-Herrera current μ can be written locally as $\mu = a \mu^{\gamma} \wedge \omega$ for such a tuple γ and some holomorphic function a , see, e.g., [2, Theorem 1.1]. Thus the annihilator

$\text{ann } \mu$ is the kernel of the sheaf mapping $\mathcal{O} \rightarrow \mathcal{O}/(\gamma)$, $\phi \mapsto a\phi$, and hence $\text{ann } \mu$ is coherent.

Let $S \rightarrow Y$ be a vector bundle. We say that $b \in \text{ASM}(Y, S)$ if there is a representation (2.5), where ω is a smooth section of $L \otimes \pi^* S$. The statements above have analogues for S -valued sections. For instance, if S is a line bundle and $\gamma_j \in \text{ASM}(Y, S)$, then (2.9) is an S^{-p} -valued current.

3. GLOBAL COLEFF-HERRERA CURRENTS ON \mathbb{P}^N

Let δ_x be interior multiplication by the vector field

$$\sum_1^N x_j \frac{\partial}{\partial x_j}$$

on \mathbb{C}^{N+1} and recall that a differential form ξ on $\mathbb{C}^{N+1} \setminus \{0\}$ is projective, i.e., the pullback of a form on \mathbb{P}^N , if and only if $\delta_x \xi = \delta_{\bar{x}} \xi = 0$, where $\delta_{\bar{x}}$ is the conjugate of δ_x . We will identify forms on \mathbb{P}^N and projective forms. Notice that

$$\Omega = \delta_x(dx_0 \wedge \dots \wedge dx_N)$$

is a non-vanishing section of the trivial bundle over \mathbb{P}^N , realized as a $(N, 0)$ -form on \mathbb{P}^N with values in $\mathcal{O}(N+1)$.

Let $\gamma_1, \dots, \gamma_p$ be holomorphic sections of $\mathcal{O}(r)$ such that their common zero set Z_γ has codimension p . Then, cf., Section 2.4 above,

$$(3.1) \quad \mu^\gamma \wedge \Omega = \bar{\partial} \frac{1}{\gamma_p} \wedge \dots \wedge \bar{\partial} \frac{1}{\gamma_1} \wedge \Omega$$

is a global section of $\mathcal{CH}^{Z_\gamma} \otimes \mathcal{O}(-pr + N + 1)$.

Lemma 3.1. *Let $Z \subset Z_\gamma$ be a reduced projective variety of pure codimension p and let μ be a global section of $\mathcal{CH}^Z \otimes \mathcal{O}(\ell + N + 1)$ such that*

$$(3.2) \quad \gamma_1 \mu = \dots = \gamma_p \mu = 0.$$

If $p \leq N - 1$, then there is a global holomorphic section a of $\mathcal{O}(\ell + pr)$ such that

$$(3.3) \quad \mu = a \bar{\partial} \frac{1}{\gamma_p} \wedge \dots \wedge \bar{\partial} \frac{1}{\gamma_1} \wedge \Omega.$$

If $p = N$ and $\ell + N \geq 0$, then the same conclusion holds.

In particular we see that if $p \leq N - 1$ and $\ell + pr < 0$, then $\mu = 0$.

Proof. Let us introduce a trivial vector bundle E of rank p with global holomorphic frame elements e_1, \dots, e_p and let e_1^*, \dots, e_p^* be the dual frame for E^* . We then have the mapping interior multiplication $\delta_\gamma: \Lambda^{*+1} E \rightarrow \Lambda^* E$ by the section $\gamma := \gamma_1 e_1^* + \dots + \gamma_p e_p^*$ of E^* . We consider the exterior algebra of $E \oplus T^* \mathbb{P}^N$ so that $d\bar{x}_j \wedge e_j = -e_j \wedge d\bar{x}_j$ etc. Then both δ_γ and $\bar{\partial}$ extend to mappings on currents with values in ΛE , and

$$(3.4) \quad \delta_\gamma \bar{\partial} = -\bar{\partial} \delta_\gamma.$$

Let $e = e_1 \wedge \dots \wedge e_p$. Recall that $H^{N,k}(\mathbb{P}^N, \mathcal{O}(\nu)) = 0$ if either $1 \leq k \leq N - 1$ or $k = N$ and $\nu \geq 1$; see, e.g., [12, Ch. VII, Theorem 10.7]. If $p \leq N - 1$, or $\ell + N + 1 \geq 1$, we can therefore find a global solution to $\bar{\partial} w_{p-1} = \mu \wedge e$. In view of (3.4) and (3.2) we have that

$$\bar{\partial} \delta_\gamma w_{p-1} = -\delta_\gamma \bar{\partial} w_{p-1} = -\delta_\gamma(\mu \wedge e) = 0.$$

Thus we can successively solve

$$(3.5) \quad \bar{\partial}w_{p-1} = \mu \wedge e, \quad \bar{\partial}w_{p-2} = \delta_\gamma w_{p-1}, \dots, \bar{\partial}w_0 = \delta_\gamma w_1.$$

Then $a \wedge \Omega := \delta_\gamma w_0$ is a $\bar{\partial}$ -closed, and thus a holomorphic, $(N, 0)$ -form with values in $\mathcal{O}(\ell + pr + N + 1)$. Altogether,

$$(\delta_\gamma - \bar{\partial})w = a \wedge \Omega - \mu \wedge e$$

if $w = w_0 + \dots + w_{p-1}$. As in [2, Examples 3.1 or 3.2] we can find a global current U such that

$$(\delta_\gamma - \bar{\partial})U = 1 - \mu^\gamma \wedge e.$$

Thus

$$(\delta_\gamma - \bar{\partial})(aU \wedge \Omega - w) = \mu - a\mu^\gamma \wedge \Omega.$$

Since the right hand side is in \mathcal{CH}^Z it now follows from [2, Theorem 3.3] that it must vanish. \square

Example 3.2. Given a global section μ of $\mathcal{CH}_Z \otimes \mathcal{O}(\ell)$ one can always find γ_j such that (3.2) holds. In fact, for a large enough r_0 there are sections g'_1, \dots, g'_m of $\mathcal{O}(r_0)$ that generate the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^N}$. If g_1, \dots, g_p are generic linear combinations of the g'_j , then $Z_g = \{g_1 = \dots = g_p = 0\}$ has codimension p , $Z_g \supset Z$, and (expressed in a local frame) $dg_1 \wedge \dots \wedge dg_p \neq 0$ on Z_{reg} . If $\gamma_j = g_j^{\mathfrak{m}_j+1}$ and \mathfrak{m}_j are large enough, then (3.2) holds. \square

4. BJÖRK-TYPE REPRESENTATION OF GLOBAL COLEFF-HERRERA CURRENTS

In this section we express the action $\mu.\xi$ if a global Coleff-Herrera current μ on a test form ξ as an integral over Z of $\mathcal{M}\xi$, where \mathcal{M} is a certain differential operator.

As usual we identify smooth sections ψ of the line bundle $\mathcal{O}(\ell)$ by ℓ -homogeneous smooth functions on $\mathbb{C}^{N+1} \setminus \{0\}$. Notice that then each $\partial/\partial x_j$, $j = 0, \dots, N$, induces a differential operator $\mathcal{O}(\ell) \rightarrow \mathcal{O}(\ell - 1)$. We say that a finite sum

$$(4.1) \quad \mathbf{L} = \sum_{\alpha} v_{\alpha} \frac{\partial^{\alpha}}{\partial x^{\alpha}}$$

is a holomorphic differential operator on \mathbb{P}^N of *degree* r if the coefficients v_{α} are holomorphic sections of $\mathcal{O}(r + |\alpha|)$. Such an \mathbf{L} maps $\mathcal{O}(\ell) \rightarrow \mathcal{O}(\ell + r)$ for each ℓ . The *order* of \mathbf{L} is the maximal occurring $|\alpha|$ as usual.

Consider the affinization $\mathbb{C}^N \simeq \{x_0 \neq 0\}$. Notice that there is a one-to-one correspondence between smooth sections of $\mathcal{O}(\ell)$ over \mathbb{C}^N and smooth functions in \mathbb{C}^N , via the frame $[x_0, \dots, x_N] \mapsto x_0^{\ell}$ for $\mathcal{O}(\ell)$ over \mathbb{C}^N . More concretely, given the section ϕ one gets the associated function by just letting $x_0 = 1$. Conversely, given Φ , then $\phi(x) = x_0^{\ell} \Phi(x'/x_0)$. In this way a differential operator of degree r gives rise to a differential operator

$$L = \sum_{|\alpha'| \leq M} V_{\alpha'}(x') \frac{\partial^{\alpha'}}{\partial x^{\alpha'}}$$

where $V_{\alpha'}(x')$ are polynomials of degree at most $r + |\alpha'|$. Notice however, that the resulting affine L will depend on ℓ unless $\mathbf{L}(x_0 \phi) = x_0 \mathbf{L} \phi$ for all ϕ . For instance, the differential operator $\mathbf{L} = \partial/\partial x_0$, that has order 1 and degree -1 , induces

$$L = \ell - \sum_{j=1}^N x_j \frac{\partial^j}{\partial x^j}.$$

Notice that L , as well as an associated affine differential operator L , act on smooth $(0, *)$ -forms as well.

The following statement is a global version of a construction due to Björk, [10]. A similar result is obtained in [29, Theorem 4.2].

Theorem 4.1. *Assume that $Z \subset \mathbb{P}^N$ has pure codimension p , that μ is a global section of $\mathcal{CH}_Z \otimes \mathcal{O}(r)$, and assume that $p \leq N - 1$ or $r + 1 \geq 0$. Let $\mathcal{I} = \text{ann } \mu$. There is a multiindex $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_p)$, a number ρ , and for each $\alpha \leq \mathbf{m}$ there are holomorphic differential operators L_α and $\mathcal{M}_{\mathbf{m}-\alpha}$, such that $\deg L_\alpha + \deg \mathcal{M}_{\mathbf{m}-\alpha} = \rho$, and a global meromorphic $(n, 0)$ -form τ with values in $\mathcal{O}(-\rho)$, not identically polar on any irreducible component of Z , such that the following hold:*

(i) *For any global holomorphic section ϕ of $\mathcal{O}(\ell)$ and any test form ξ of bidegree $(0, n)$ with values in $\mathcal{O}(-r - \ell)$ we have*

$$(4.2) \quad \phi \mu \cdot \xi = \sum_{\alpha \leq \mathbf{m}} \int_Z \tau \wedge L_\alpha \phi \wedge \mathcal{M}_{\mathbf{m}-\alpha} \xi.$$

(ii) *For each point $x \in Z$, a germ $\psi \in \mathcal{O}_x$ is in \mathcal{I}_x if and only if*

$$(4.3) \quad L_\alpha \psi \in \sqrt{\mathcal{I}_x}, \quad \alpha \leq \mathbf{m}.$$

(iii) *For each $\alpha \leq \mathbf{m}$ there are holomorphic differential operators $\mathcal{M}_{\alpha, \gamma}$, $\gamma \leq \alpha$, such that*

$$(4.4) \quad L_\alpha(\phi\psi) = \sum_{\gamma \leq \alpha} L_\gamma \phi \mathcal{M}_{\alpha, \gamma} \psi$$

for all holomorphic sections ϕ and ψ of $\mathcal{O}(\ell)$ and $\mathcal{O}(\ell')$.

Proof. To begin with we choose g_1, \dots, g_p , $\mathbf{m} := (\mathbf{m}_1, \dots, \mathbf{m}_p)$, and a as in Example 3.2 and Lemma 3.1 so that

$$(4.5) \quad \mu = a \mu^{g^{\mathbf{m}+1}} \wedge \Omega.$$

After a projective transformation on \mathbb{P}^N , i.e., a linear change of variables on \mathbb{C}^{N+1} , we may assume that each irreducible component of Z intersects the affine space $\mathbb{C}^N := \{x_0 \neq 0\}$. Then the affinizations G_j of g_j are polynomials in \mathbb{C}^N such that $dG_1 \wedge \dots \wedge dG_p$ is nonvanishing on $Z_{\text{reg}} \cap \mathbb{C}^N$, cf., Example 3.2. Let $x' = (x_1, \dots, x_N)$. After possibly a linear transformation of \mathbb{C}^N , we may assume that the polynomial

$$H := \det \frac{\partial G}{\partial \eta}$$

is generically nonvanishing on $Z \cap \mathbb{C}^N$, where

$$x' = (\zeta, \eta) = (\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_p).$$

Let us introduce the short hand notation

$$\bar{\partial} \frac{1}{G^{\mathbf{m}+1}} = \bar{\partial} \frac{1}{G_1^{\mathbf{m}_1+1}} \wedge \dots \wedge \bar{\partial} \frac{1}{G_p^{\mathbf{m}_p+1}}.$$

We first look for a representation of the Coleff-Herrera current

$$\tilde{\mu} = \bar{\partial} \frac{1}{G^{\mathbf{m}+1}} \wedge d\eta \wedge d\zeta$$

at points x on $Z' := Z \cap \mathbb{C}^N \cap \{H \neq 0\}$. Locally at such a point we can make the change of variables

$$w = G(\zeta, \eta), \quad z = \zeta.$$

If Ξ is a smooth $(0, n)$ -form with small support, and Φ is holomorphic, with the notation $m! = m_1! \cdots m_p!$ and $\partial_w^\alpha = \partial^{|\alpha|}/\partial w^\alpha$, etc, in view of Example 2.1 we then have

$$\begin{aligned} \Phi \tilde{\mu} \cdot \Xi &= \int \bar{\partial} \frac{1}{G^{\mathbf{m}+1}} \wedge d\eta \wedge d\zeta \wedge \Phi \Xi = \pm \int \bar{\partial} \frac{1}{w^{\mathbf{m}+1}} \wedge dw \wedge dz \wedge \frac{\Xi}{H} \Phi = \\ &\pm \int_{w=0} \frac{(2\pi i)^p}{\mathbf{m}!} dz \wedge \partial_w^{\mathbf{m}} \left(\frac{\Xi}{H} \Phi \right) = \pm \sum_{\alpha \leq \mathbf{m}} \int_{w=0} \frac{(2\pi i)^p}{(\mathbf{m} - \alpha)! \alpha!} dz \wedge \partial_w^{\mathbf{m} - \alpha} \left(\frac{\Xi}{H} \right) \partial_w^\alpha \Phi. \end{aligned}$$

Now, notice that

$$\partial_\eta = (\partial_\eta G) \partial_w$$

so that

$$\partial_w = \frac{\Gamma}{H} \partial_\eta,$$

where Γ is a matrix of polynomials. It is readily checked that

$$(4.6) \quad \tilde{L}_\alpha := H^{2|\alpha|} \left(\frac{\Gamma}{H} \partial_\eta \right)^\alpha$$

has a holomorphic extension across $H = 0$. Let us define

$$M_\beta \Xi = \pm \frac{(2\pi i)^p}{\beta! (\mathbf{m} - \beta)!} H^{1+|\mathbf{m}|+2|\beta|} \left(\frac{\Gamma}{H} \partial_\eta \right)^\beta \frac{\Xi}{H}.$$

Then also M_β is holomorphic across $H = 0$.

With $T = dz = d\zeta$, we have that

$$(4.7) \quad \Phi \tilde{\mu} \cdot \Xi = \int_{Z'} \sum_{\alpha \leq \mathbf{m}} \frac{T}{H^{3|\mathbf{m}|+1}} \wedge M_{\mathbf{m}-\alpha} \Xi \wedge \tilde{L}_\alpha \Phi$$

for Ξ with support close to x . We claim that if Φ is a germ of a holomorphic function at x , then $\Phi \tilde{\mu}_x = 0$ if and only if $\tilde{L}_\alpha \Phi = 0$ on Z_x for all $\alpha \leq \mathbf{m}$. In fact,

$$(4.8) \quad \begin{aligned} \Phi \tilde{\mu}_x = 0 &\iff \Phi \bar{\partial} \frac{1}{G^{\mathbf{m}+1}}|_x = 0 \iff \Phi \bar{\partial} \frac{1}{w^{\mathbf{m}+1}}|_x = 0 \iff \\ &\partial_w^\alpha \Phi = 0 \text{ on } Z_x, \alpha \leq \mathbf{m} \iff \tilde{L}_\alpha \Phi = 0 \text{ on } Z_x, \alpha \leq \mathbf{m}. \end{aligned}$$

Now, for each $\alpha \leq \mathbf{m}$, let us homogenize the coefficients in \tilde{L}_α to obtain \tilde{L}_α for some fixed degree, and then let us homogenize $M_{\mathbf{m}-\alpha}$ to $\mathcal{M}_{\mathbf{m}-\alpha}$ so that the sum of their degrees is a fixed number ρ . Let τ' be the homogenization of $T = d\zeta$, i.e.,

$$\tau' = d \frac{x_1}{x_0} \wedge \cdots \wedge d \frac{x_n}{x_0}$$

if $x = (x_0, \dots, x_N) = (x_0, \zeta, \eta)$. Finally let us homogenize $H^{3|\mathbf{m}|+1}$ to h so that $\tau := \tau'/h$ takes values in $\mathcal{O}(-\rho)$. We possibly get some factors x_0 in the denominator, but since Z has no irreducible component in $\{x_0 = 0\}$ this is acceptable.

Let us define the global current

$$(4.9) \quad \tilde{\mu} := \mathbf{1}_Z \mu^{g^{\mathbf{m}+1}} \wedge \Omega$$

in \mathbb{P}^N . In view of (4.5) it takes values in $\mathcal{O}(r - \deg a)$. At each point $x \in Z'$ it is the $(r - \deg a)$ -homogenization of our previous $\tilde{\mu}$ but the global current is not necessarily $\bar{\partial}$ -closed at x_0 . However, in view of (4.5), (2.7), and (4.9),

$$(4.10) \quad a\tilde{\mu} = a\mathbf{1}_Z \mu^{g^{\mathbf{m}+1}} \wedge \Omega = \mathbf{1}_Z a\mu^{g^{\mathbf{m}+1}} \wedge \Omega = \mathbf{1}_Z \mu = \mu,$$

since μ has support on Z , and thus $a\tilde{\mu}$ is $\bar{\partial}$ -closed.

For holomorphic sections ϕ of $\mathcal{O}(\ell - \deg a)$ and test forms ξ of bidegree $(0, n)$ with support in $\mathbb{P}^N \setminus \{h = 0, x_0 = 0\}$ and values in $\mathcal{O}(-r - \ell)$ we have

$$(4.11) \quad \phi \tilde{\mu} \cdot \xi = \int_Z \sum_{\alpha \leq \mathfrak{m}} \tau \wedge \mathcal{M}_{\mathfrak{m}-\alpha} \xi \wedge \tilde{\mathbf{L}}_\alpha \phi.$$

By Proposition 2.6, $\tau \wedge \tilde{\mathbf{L}} \phi \wedge [Z]$ is a global section of $\mathcal{W}^Z \otimes \mathcal{O}(e + \ell)$ and thus the integrals on the right hand side of (4.11) exist as principal values for any test form ξ . In view of Proposition 2.4 the right hand side of (4.11) defines the action on ξ of a global section of $\mathcal{W}^Z \otimes \mathcal{O}(e + \ell)$. Since $\{h = 0, x_0 = 0\} \cap Z$ has positive codimension on Z it follows by the SEP that the equality (4.11) holds for all ξ .

Define the holomorphic differential operators \mathbf{L}_α by the equality

$$(4.12) \quad \mathbf{L}_\alpha \phi = \tilde{\mathbf{L}}_\alpha(a\phi).$$

Then (4.2) follows from (4.11). Thus (i) is proved.

For $x \in Z' = Z \setminus \{h = 0, x_0 = 0\}$ we have, by (4.8) and (4.12), that

$$(4.13) \quad \phi \mu_x = 0 \text{ if and only if } \mathbf{L}_\alpha \phi = 0 \text{ on } Z_x, \alpha \leq \mathfrak{m}.$$

Again since $\{h = 0, x_0 = 0\} \cap Z$ has positive codimension on Z , it follows by continuity and the SEP that (4.13) holds for all $x \in Z$. Thus (ii) is proved.

To see (iii), just notice that

$$\tilde{\mathbf{L}}_\alpha(\Phi\Psi) = \sum_{\gamma \leq \alpha} L_\gamma \Phi c_{\alpha, \gamma} L_{\alpha-\gamma} \Psi,$$

where $c_{\alpha, \gamma}$ are binomial coefficients. After homogenization and replacing ϕ by $a\phi$ we get (iii) with $\mathbf{L}_{\alpha, \gamma} = c_{\alpha, \gamma} \mathbf{L}_{\alpha-\gamma}$. \square

Remark 4.2. One can check, cf., [6, Section 5], that $\phi \mathbf{1}_Z \tilde{\mu} = 0$ if and only if ϕ is in the intersection of the primary ideals of $(g^{\mathfrak{m}+1})$ associated with the irreducible components of Z . \square

Let μ be a global section of $\mathcal{CH}^Z \otimes \mathcal{O}(r)$ in \mathbb{P}^N and let b be a global almost semi-meromorphic current of bidegree $(0, *)$ with values in $\mathcal{O}(r_1)$. Then $b\mu$ is a section of $\mathcal{W}^Z \otimes \mathcal{O}(r + r_1)$. Let us also assume that $ZSS(b) \cap Z$ has positive codimension in Z . Consider a representation of μ as in Theorem 4.1. In view of Theorem 2.5 we can define differential operators \widehat{M}_γ with almost semi-meromorphic coefficients so that

$$\widehat{M}_\gamma \xi = M_\gamma(b\xi).$$

For test forms ξ of bidegree $(0, *)$ with values in $\mathcal{O}(-r - \ell)$ and with support outside $ZSS(b)$, and any global holomorphic section ϕ of $\mathcal{O}(\ell)$ we have

$$(4.14) \quad \phi b \mu \cdot \xi = \sum_{\alpha \leq \mathfrak{m}} \int_Z \tau \wedge \mathbf{L}_\alpha \phi \wedge \widehat{M}_{\mathfrak{m}-\alpha} \xi.$$

In view of Propositions 2.6 and 2.4 the right hand side defines a global section of $\mathcal{W}^Z \otimes \mathcal{O}(r + r_1)$. Since $Z \cap ZSS(b)$ has positive codimension in Z , it follows that (4.14) holds globally.

5. PROOF OF THEOREM 1.2

Let X be our non-reduced subspace of \mathbb{P}^N . As was mentioned in the introduction the proof relies on the global current $R^f \wedge R^X$ that we first discuss.

5.1. The current R^X . Given a vector bundle $E \rightarrow \mathbb{P}^N$, let $\mathcal{O}(E)$ denote the associated locally free analytic sheaf. We can find a locally free resolution

$$0 \rightarrow \mathcal{O}(E_N) \xrightarrow{c_N} \cdots \xrightarrow{c_2} \mathcal{O}(E_1) \xrightarrow{c_1} \mathcal{O}(E_0) \rightarrow \mathcal{O}_{\mathbb{P}^N}/\mathcal{J}_X \rightarrow 0$$

of $\mathcal{O}_{\mathbb{P}^N}/\mathcal{J}_X$, where E_0 is a trivial line bundle and $E_k = \bigoplus_i^{r_k} \mathcal{O}(-d_k^i)$ for suitable positive numbers d_k^i , see, e.g., [7]. In fact, we can use the "same" mappings $c_k = (c_k^{ij})$ as in (2.1) but with c_k^{ij} considered as sections of $\mathcal{O}(d_k^j - d_{k-1}^i)$. There is a natural choice of Hermitian metrics on E_k and following [5, Sections 3 and 6] there is an associated current

$$R^X = R_p^X + \cdots + R_N^X$$

with support on X_{red} , where R_k^X are $(0, k)$ -currents that take values in E_k , and with the property that $\phi R^X = 0$ if and only if $\phi \in \mathcal{J}_X$. Furthermore,

$$(5.1) \quad \bar{\partial} R_k^X = c_{k+1} R_{k+1}^X, \quad k \geq 0.$$

Proposition 5.1. *There is a bundle*

$$(5.2) \quad F = \bigoplus_{i=1}^{r_F} \mathcal{O}(d_F),$$

a global section μ of $\mathcal{CH}^{X_{red}} \otimes F \otimes \mathcal{O}(N+1)$, and an almost semi-meromorphic section b of $\text{Hom}(F, \bigoplus_{i=p}^{N+1} E_k)$ such that

$$(5.3) \quad R^X \wedge \Omega = b\mu.$$

in \mathbb{P}^N .

Proof. Since the kernel \mathcal{K} of $c_{p+1}^*: \mathcal{O}(E_p^*) \rightarrow \mathcal{O}(E_{p+1}^*)$ is coherent, for a large enough integer d_F , $\mathcal{K} \otimes \mathcal{O}(d_F)$ is generated by global sections g_1, \dots, g_{r_F} . We therefore have a surjective sheaf mapping $\bigoplus_1^{r_F} \mathcal{O} \rightarrow \mathcal{K} \otimes \mathcal{O}(d_F)$ and hence $\bigoplus_1^{r_F} \mathcal{O}(-d_F) \rightarrow \mathcal{K}$. Define F by (5.2) and let $g: \mathcal{O}(E_p) \rightarrow \mathcal{O}(F)$ be the dual of the composed mapping $\mathcal{O}(F^*) \rightarrow \mathcal{K} \rightarrow \mathcal{O}(E_p^*)$. We then have the exact sequence

$$(5.4) \quad \mathcal{O}(F^*) \xrightarrow{g^*} \mathcal{O}(E_p^*) \xrightarrow{c_{p+1}^*} \mathcal{O}(E_{p+1}^*)$$

of sheaves. We claim that

$$\mu := gR_p^X \wedge \Omega$$

is a global (vector-valued) Coleff-Herrera current. In fact, in view of (5.1),

$$\bar{\partial}\mu = \bar{\partial}gR_p^X \wedge \Omega = g\bar{\partial}R_p^X \wedge \Omega = gc_{p+1}R_{p+1}^X \wedge \Omega = 0,$$

since $gc_{p+1} = 0$. Because of the dimension principle μ must have the SEP with respect to X_{red} and hence it is, by definition, a Coleff-Herrera current and thus a section of $\mathcal{CH}^{X_{red}} \otimes F \otimes \mathcal{O}(N+1)$.

Let X_{p+1} be the subset of X_{red} where s_{p+1} does not have optimal rank. Let us choose a Hermitian norm on F , and define $\sigma_F: F \rightarrow E_p$ on the complement of X_{p+1} so that $\sigma_F = 0$ on the orthogonal complement of $\text{Im } g$ and $\sigma_F g = I$ on the orthogonal complement of $\text{Ker } g$. It is shown in [6, Section 2] that σ_F has an almost semi-meromorphic extension across X_{p+1} ; let us denote the extension by σ_F as well. Following the proof of [28, Proposition 3.2] we see (this is just a local argument) that $R_p^X = \sigma_F g R_p^X$ outside X_{p+1} . The right hand side here is defined in view of Theorem 2.6. Since both sides have the SEP on X_{red} we conclude that they coincide in \mathbb{P}^N . Thus

$$(5.5) \quad R_p^X \wedge \Omega = \sigma_F \mu.$$

From [5, Theorem 4.4] we get global almost semi-meromorphic sections α_{k+1} of $\text{Hom}(E_k, E_{k+1})$, $k = p, p+1, \dots$, that are smooth outside analytic subsets X_{k+1} of X_{red} where s_{k+1} do not have optimal rank, such that

$$R_{k+1}^X = \alpha_{k+1} R_k^X.$$

Since X has pure dimension it follows that $\text{codim } X_{p+\ell} \geq p + \ell + 1$ according to [14, Corollary 20.14]. Arguing as in the proof of [28, Proposition 3.2] we now get for each $k \geq p+1$, in view of (5.5), the representation

$$(5.6) \quad R_k^X = \alpha_k \cdots \alpha_{p+1} \sigma_F \mu.$$

Now let $b_k = \alpha_k \cdots \alpha_{p+1} \sigma_F$. Then b_k is an almost semi-meromorphic, see [8, Section 3.1], and by (5.6), $R_k^X = b_k \mu$ where b_k is smooth, that is, outside Z_{p+1} . Since $\mathbf{1}_{Z_{p+1}} \mu = 0$ it follows from (2.7) that $R_k^X = b_k \mu$. Thus the proposition follows with $b = b_p + \cdots + b_N$. \square

5.2. The current $R^a \wedge R^X$. Assume that we have sections a_1, \dots, a_m of a Hermitian line bundle S over some open set $\mathcal{U} \subset \mathbb{P}^N$ and let E be a trivial rank m bundle. Then we have interior multiplication $\delta_a: \Lambda^{*+1} E \otimes S^{-*-1} \rightarrow \Lambda^* E \otimes S^{-*}$, and we can consider the induced double complex as in the proof of Lemma 3.1 above. Following [7, Example 2.1] we define the Bochner-Martinelli form $U^a = U_1^a + \cdots + U_N^a$, explicitly from the a_j . The components U_k^a are almost semi-meromorphic $(0, k-1)$ -forms with values in $\Lambda^k E \otimes S^{-k}$ that are smooth outside the common zero set Z_a of the a_j . Moreover, $(\delta_a - \bar{\partial})U^a = 1$ outside Z_a . We thus have the residue current

$$R^a := 1 - (\delta_a - \bar{\partial})U^a,$$

with support on Z_a , whose components R_k^a are $(0, k)$ -currents with values in $\Lambda^k E \otimes S^{-k}$. If $\chi_\epsilon = \chi(|a|^2/\epsilon)$, where χ is a function as in (2.4) above, then $U^{a,\epsilon} = \chi_\epsilon U^a$ are smooth and tend to U^a . Thus

$$R^{a,\epsilon} = 1 - (\delta_a - \bar{\partial})U^{a,\epsilon} = 1 - \chi_\epsilon + \bar{\partial}\chi_\epsilon \wedge U^a$$

tend to R^a . As in [7, Section 2.5], cf., (2.8) above, we can form the product

$$(5.7) \quad R^a \wedge R^X \wedge \Omega := \lim_{\epsilon \rightarrow 0} R^{a,\epsilon} \wedge R^X \wedge \Omega.$$

We will use the following important property, which follows from [7, (2.19)] and the proof [7, Lemma 2.2]:

Lemma 5.2. *If Φ is holomorphic and $\Phi R^a \wedge R^X \wedge \Omega = 0$ at x , then Φ is in $(a)_x + \mathcal{I}_{X,x}$.*

Remark 5.3 (Warning!). Although the components R_k^a of R^a vanish for small k because of the dimension principle, the terms $R_k^a \wedge R^X$ might be nonzero. See, e.g., [8] for examples. \square

5.3. End of proof of Theorem 1.2. To begin with we assume that $p = \text{codim } Z \leq N - 1$. Let μ be the (vector-valued) Coleff-Herrera current in the representation (5.3) of $R^X \wedge \Omega$. Let us consider μ as an r_F -tuple of Coleff-Herrera currents, and let \mathbf{L}_α , $\alpha \leq \mathbf{m}$, be a (tuple of) Noetherian operators obtained from Theorem 4.1. Moreover, let \widehat{M}_α be the associated differential operators with almost semi-meromorphic coefficients so that (4.14) holds.

At a given point $x \in X_{red}$ there is a number ν_x such that if $(a) = (a_1, \dots, a_m) \subset \mathcal{O}_{X,x}$ is a local ideal, and $\phi \in \mathcal{O}_{X,x}$, then $|\mathbf{L}_\alpha \phi| \leq C|a|^\nu$ on $X_{red,x}$ for all $\alpha \leq \mathbf{m}$ implies that $\phi R^a \wedge R^X \wedge \Omega = 0$. This is precisely the main step of the proof of [28,

Theorem 1.2] and we do not repeat it here (just notice that our number ν_x is called N in [28], our \widetilde{M}_α are called \widetilde{K}_α , moreover, the non-reduced space that we call X is denoted by Z in [28] whereas X denotes the associated reduced space!). In this proof the number ν_x is explicitly deduced from the singularities of the coefficients of \widetilde{M}_α and of b , expressed as the degree of monomials in a suitable log resolution of X_{red} , see [28, Eq. (4.9)]. In particular, the number ν_x works for all points in a neighborhood of x . By compactness we therefore get:

Proposition 5.4. *There is a number ν , such that if $x \in X_{red}$, $(a) = (a_1, \dots, a_m) \subset \mathcal{O}_{X,x}$ is a local ideal, and $\phi \in \mathcal{O}_{X,x}$, then $|L_\alpha \phi| \leq C|a|^\nu$ on $X_{red,x}$ for all $\alpha \leq \mathbf{m}$ implies that $\phi R^a \wedge R^X \wedge \Omega = 0$.*

Combined with Lemma 5.2 we have thus obtained ν and differential operators L_α so that part (i) of Theorem 1.2 holds.

Now let F_j be polynomials as in Theorem 1.2 (ii), let f_j be the d -homogenizations considered as section of $\mathcal{O}(d)$ over X_{red} and let \mathcal{J}_f be the associated ideal sheaf as in the introduction.

Lemma 5.5. *Let Φ be a polynomial such that (1.4) holds and let ϕ be the ρ -homogenization of Φ . If*

$$(5.8) \quad \rho \geq \deg \Phi + \nu d^\infty \deg X_{red},$$

then $|L_\alpha \phi| \leq C|f|^\nu$ for all α .

Proof. Let $\pi: \tilde{X} \rightarrow X_{red}$ be the normalization of the blow-up of X_{red} along \mathcal{J}_f and let $\sum r_j W_j$ be the exceptional divisor, where W_j are the irreducible components and r_j the corresponding multiplicities. Notice that if ψ is a holomorphic section of some $\mathcal{O}(\ell)$, then $|\psi| \leq C|f|^\nu$ if and only if $\pi^* \psi$ vanishes to order at least νr_j on W_j for each j .

If (1.4) holds on V_{red} , then $\pi^*(L_\alpha \phi)$ vanishes to order νr_j on each W_j that is not fully contained in $\pi^{-1}(X_{red,\infty})$. Notice that

$$\phi = x_0^{\rho - \deg \Phi} \varphi,$$

where φ is the $\deg \Phi$ -homogenization of Φ and thus holomorphic. If W_j is contained in $\pi^{-1}X_{red,\infty}$, then ϕ vanishes at least to order $\rho - \deg \Phi$ on W_j . Since $L_\alpha \phi$ does not involve the derivative $\partial/\partial x_0$ also $L_\alpha \phi$ vanishes to order $\rho - \deg \Phi$ on W_j . By the geometric estimate in [13], cf., [7, Eq. (6.2)], we have that

$$r_j \leq d^{\text{codim } \pi(W_j)} \deg X_{red}.$$

If (5.8) holds, therefore $\pi^*(L_\alpha \phi)$ vanishes, at least, to order νr_j on W_j for all j . Thus the lemma follows. \square

With the same hypotheses as in Lemma 5.5 it follows from the lemma and Proposition 5.4 that

$$(5.9) \quad \phi R^f \wedge R^X \wedge \Omega = 0.$$

If in addition

$$\rho \geq (d-1) \min(m, n+1) + \text{reg } X,$$

we can now solve a sequence of global $\bar{\partial}$ -equations in \mathbb{P}^N and get a global solution q_j to $\phi = f_1 q_1 + \dots + f_m q_m$, cf., [7, Lemma 4.3]. The fact that X is not reduced plays no role here. After dehomogenization we obtain the desired representation of Φ , and so the proof of Theorem 1.2 is complete in case $p \leq N-1$.

Now assume that $p = \text{codim } Z = N$ so that X_{red} is a finite set in $\mathbb{C}^N \simeq \mathbb{P}^N \setminus \{x_0 = 0\}$. If necessary we multiply μ by a suitable power of x_0 to be able to apply Theorem 4.1. We then get the global, in \mathbb{C}^N , L_α that form a complete set of Noetherian operators at each point $x \in X_{\text{red}}$. Part (ii) is trivial, since the image of any ideal $(a) \subset \mathcal{O}_{X,x}$ in $\mathcal{O}_{X_{\text{red}},x}$ is just either (0) or $(1) = \mathcal{O}_{X_{\text{red}},x}$.

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